The 11th International Conference on Applied Informatics
*to be held in Eger, Hungary*
*January 29–31, 2020*

# Representing Directed Graphs as 3-SAT Problems using the Simplified Balatonboglár Model

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## Abstract

In our previous work we introduced the **Balatonboglár Model** ($\mathcal{BB}$) of directed graphs. We showed that $\mathcal{BB}$ is a `Black-and-White 3-SAT` problem if and only if *(iff)* the graph is strongly connected (`SC`). $\mathcal{BB}$ generates a lot of so called $\mathcal{NNP}$ shaped clauses to represent cycles of the directed graph without detecting cycles, i.e., it is fast but bigger than necessary. To overcome this problem, we introduce the **Simplified Balatonboglár Model** ($\mathcal{SBB}$). The size of $\mathcal{SBB}$ is only 1% of the size of $\mathcal{BB}$.

## Introduction

In logic the most natural representation of an edge of a directed graph, say $a \to b$, is to use implication, i.e., $a \implies b$, i.e., the edge $a \to b$ can be represented by the binary clause: $(\neg a \vee b)$. If a graph contains two edges: $a \to b$, and $a \to c$, then those can be represented by the formula: $(a \implies b) \wedge (a \implies c)$, which is equivalent to two 2-clauses $(\neg a \vee b) \wedge (\neg a \vee c)$. We call this as the **Strong Model** ($\mathcal{SM}$) of directed graphs [1, 2, 5].

Our second model is the **Weak Model** ($\mathcal{WM}$) [5]. The idea is the following: If a graph contains two edges: $a \to b$, and $a \to c$, then those can be represented by the formula: $(a \implies b) \vee (a \implies c)$, which is equivalent to a 3-clause $(\neg a \vee b \vee c)$. We need to represent cycles of the graph, too. If $a_1 \to a_2 \to \cdots \to a_n \to a_1$ is a cycle with exit points $b_1, b_2, \ldots, b_m$, then this cycle can be represented by the clause: $(\neg a_1 \vee \neg a_2 \vee \cdots \vee \neg a_n \vee b_1 \vee b_2 \vee \cdots \vee b_m)$.

Our third model, the $\mathcal{BB}$ uses the trick that instead of detecting each cycle, it generates from each path $a \to b \to c$ the following 3-clause: $(\neg a \vee \neg b \vee c)$, which is a **Negative-Negative-Positive** ($\mathcal{NNP}$) shaped clause, or for short an clause, even if there is no cycle which contains the vertices $a$ and $b$. This simplification allows very fast `3-SAT` problem generation from a directed graph, and the `SAT` instance will be a `Black-and-White 3-SAT` *iff* the input directed graph is `SC`. On the other hand this trick generates a lot of superfluous clauses.

To overcome this problem, we introduce the $\mathcal{SBB}$. In this model we create the strongly connected components (to be short: `SCC`) of the graph [6, 7]. For each component we generate a cycle which contains all the nodes of the component. Then for each such cycle we generate $\mathcal{NNP}$ clauses along it. For example, for the cycle $(n_1, n_2, \ldots n_k)$ we generate the clauses $\{\{\neg n_1, \neg n_2, n_3\}, \ldots, \{\neg n_k, \neg n_1, n_2\}\}$.

We should also generate $\mathcal{NNP}$ clauses which link the components. After representing the big-cycles and the links between them, we can delete their edges and the opposite of those edges. The rest of the edges are "inner-edges" which may form cycles. We represent them one by one by an $\mathcal{NNP}$ clause, such that the positive literal should be a neighbour node on the big-cycle of one of the negative ones. Then we delete that cycle till their is no more one. Other parts of the model are just the same as in case of $\mathcal{BB}$.

## References

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## Acknowledgements

This research was supported by the grant EFOP-3.6.1-16-2016-00001 "Complex improvement of research capacities and services at Eszterházy Károly University"

## Theoretical Results

Let $\mathcal{D}$ be a communication graph [1]. Then:

- $\mathcal{SM}$ is a `Black-and-White 2-SAT` problem *iff* the graph $\mathcal{D}$ is `SC`.
- $\mathcal{WM}$ is a `Black-and-White SAT problem` *iff* $\mathcal{D}$ is `SC`.
- $\mathcal{SM} \geq \mathcal{WM}$, i.e., the set of solutions of $\mathcal{SM}$ is a subset of the set of solutions of $\mathcal{WM}$.
- $\mathcal{SM} > \mathcal{WM}$ *iff* $\mathcal{D}$ is not `SC`, and it has at least one node which has more than one child node.
- **Transitions Theorem**: If we have $\mathcal{SM} \geq \mathcal{MM} \geq \mathcal{WM}$, where $\mathcal{MM}$ is an arbitrary but fixed model of $\mathcal{D}$, then $\mathcal{MM}$ is a `Black-and-White SAT` problem *iff* $\mathcal{D}$ is `SC`.
- $\mathcal{SM} \geq \mathcal{BB} \geq \mathcal{WM}$, i.e., $\mathcal{BB}$ is a `Black-and-White 3-SAT` problem *iff* $\mathcal{D}$ is `SC`.
- $\mathcal{SM} \geq \mathcal{BB} \geq \mathcal{SBB} \geq \mathcal{WM}$.

  **Proof:** It is enough to show that for any clause in $\mathcal{WM}$ which represent a cycle there is a clause $\mathcal{D}$ in $\mathcal{SBB}$, such that $\mathcal{D}$ is a subset of that clause. Let $\mathcal{C}$ be a clause in $\mathcal{WM}$ such that it represents a cycle in $\mathcal{D}$ . From we know that the cycle has an exit point. From this we know that there is a big-cycle such that there exists $a$,$b$,$c$ such that $a, b \in \mathcal{C}$, and they are consecutive vertices in the cycle, and $c$ is an exit point of the cycle, and $b, c$ are consecutive vertices in the big-cycle. So $\mathcal{D} = \{\neg a, \neg b, c\}$ is a suitable choice.
- $\mathcal{SBB}$ is a `Black-and-White 3-SAT` problem *iff* $\mathcal{D}$ is `SC`.

## Examples

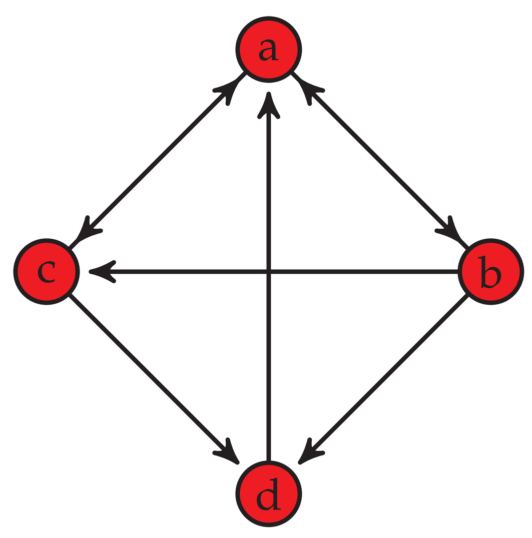



*Figure 1 - A (SCC) graph with 4 vertices, 6 cycles*

The $\mathcal{SM}$ of the $\mathcal{D}$ on Figure 1 is:

$$\mathcal{SM} = \{\{\neg a, b\}, \{\neg a, c\}, \{\neg b, a\}, \{\neg b, c\}, \{\neg b, d\}, \{\neg c, a\}, \{\neg c, d\}, \{\neg d, a\}\}. \tag{1}$$

Since the $\mathcal{D}$ on Figure is `SC`, its $\mathcal{SM}$ is a `Black-and-White SAT` problem, i.e., the `SAT` problem in (1) has only these two solutions: $\{a, b, c, d\}$, and $\{\neg a, \neg b, \neg c, \neg d\}$.

Its other 3 models are:

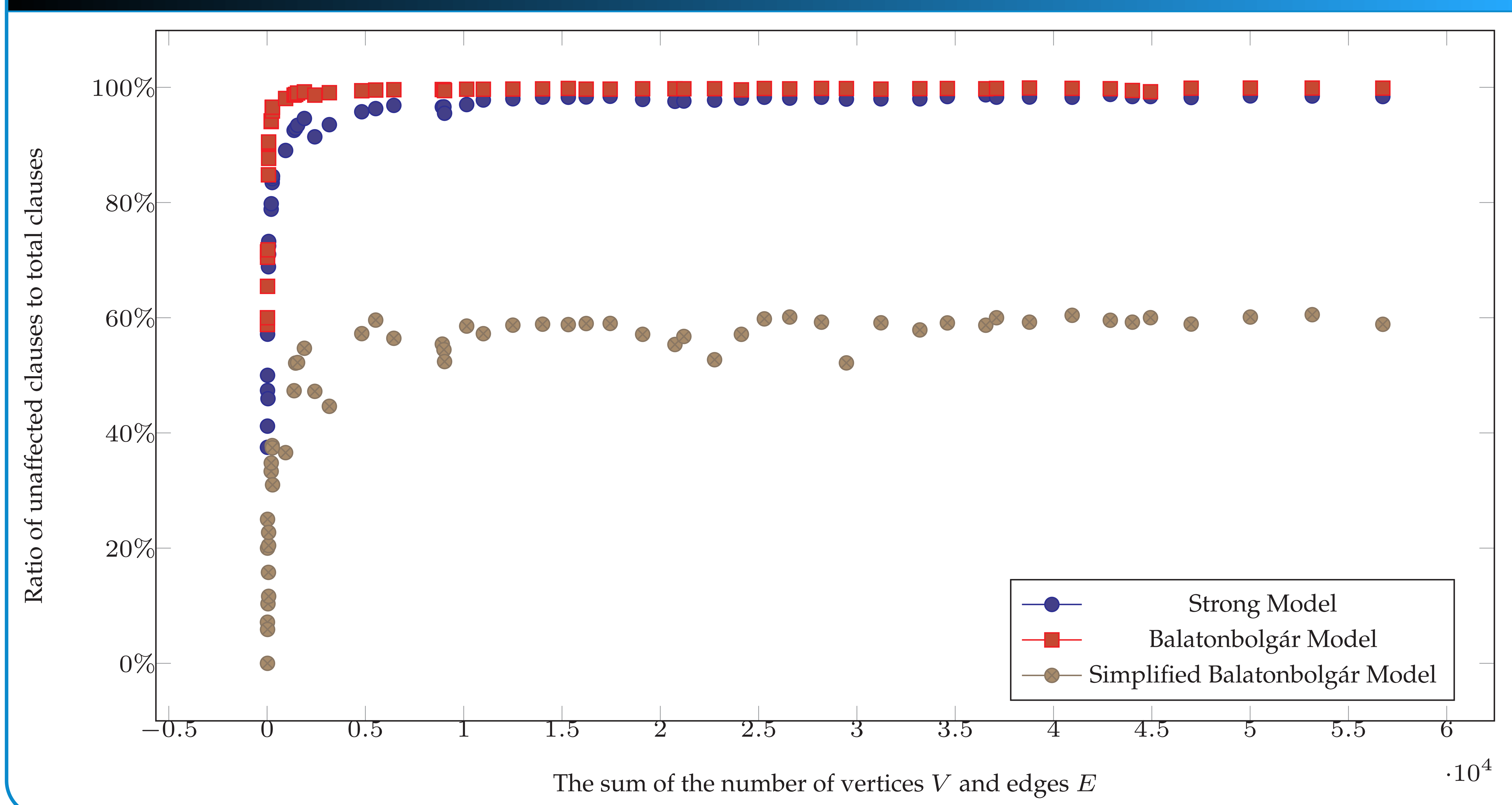
$$\mathcal{WM} = \{\{\neg a, b, c\}, \{\neg b, a, c, d\}, \{\neg c, a, d\}, \{\neg a, \neg b, c, d\}, \{\neg a, \neg b, \neg c, d\}, \{\neg a, \neg c, b, d\}, \{\neg d, a\}, \{\neg a, \neg b, \neg d, c\}, \{\neg a, \neg c, \neg d, b\}\}. \tag{2}$$

$$\mathcal{BB} = \{\{\neg a, b, c\}, \{\neg b, c, d\}, \{\neg c, a, d\}, \{\neg d, a\}, \{\neg a, \neg b, c\}, \{\neg a, \neg b, d\}, \{\neg a, \neg c, b\}, \{\neg a, \neg c, d\}, \{\neg b, \neg c, a\}, \{\neg b, \neg c, d\}, \{\neg b, \neg d, a\}, \{\neg c, \neg d, a\}, \{\neg d, \neg a, b\}, \{\neg d, \neg a, c\}\}. \tag{3}$$

$$\mathcal{SBB} = \{\{\neg a, b, c\}, \{\neg b, c, d\}, \{\neg c, a, d\}, \{\neg d, a\}, \{\neg a, \neg b, c\}, \{\neg b, \neg c, d\}, \{\neg c, \neg d, a\}, \{\neg d, \neg a, b\}, \{\neg a, \neg c, d\}\}. \tag{4}$$

All the 4 models are a `Black-and-White SAT` problem.

## Test Results




## Conclusion

Our goal was to create a more compact variant of $\mathcal{BB}$ by keeping its nice properties. The new model is called $\mathcal{SBB}$ which is a subset of $\mathcal{BB}$. Its size is around 1% of the size of $\mathcal{BB}$, but it is still a `Black-and-white 3-SAT` problem *iff* the represented directed graph is `SC`. We checked this property also empirically by using the `CSFLOC18 SAT` [3, 4] solver. Our empirical results show that $\mathcal{SBB}$ is near to the smallest possible model, i.e., the number of unaffected clauses reported by `CSFLOC18` is very small, i.e., $\mathcal{SBB}$ extended by the white and the black clause is almost `MIN-UNSAT`.