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A branching process based network evolution model describing N-interactions

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Abstract

The aim of network theory is to describe real life networks like social networks (Facebook, Twitter,...), communication networks (WWW, Internet,...), non-artificial neural networks, brain connectome, power grid, trade networks, etc. A mathematical model for a network is a random graph. A pioneering paper in network theory is the paper [3] by Barabási and Albert and a general description is the book of Barabási [2].

In this paper, we present a new continuous time network evolution model driven by a branching process. There are several papers on this approach, like [1] by Athreya, Ghosh and Sethuraman and the paper [7] by Móri and Rokob. In our former paper [5] we studied network evolution based on 3-cliques, while in [4] evolution based on 2- and 3-cliques. In the recent paper we continue the lines of [7], [5] and [4].

Now, we outline the structure of our model. The basic units are teams. Every team attracts new incomers. Teams are represented by cliques. The clique size can be $1, 2, \ldots, N$, where N is a fixed integer. At the initial time $t = 0$, we start with a single team, it can be any n-clique, $1 \leq n \leq N$. It is called the ancestor.

At certain random time a new member, i.e. a new node joins to the ancestor. So a new clique appears. Then the new clique also attracts a new member, that is a new node. So again a new clique appears and it starts its own reproduction process.

The reproduction steps of any fixed n-clique is the following. The generic n-clique has its own Poisson process $\Pi_n(t)$ with parameter 1. When the Poisson

process jumps, then a new vertex appears and it is connected to our generic n-clique. The new vertex will be connected to certain vertices of the generic n-clique. The probability that the new vertex will be connected to j vertices of the generic n-clique is $p_{n,j}$, where $0 \leq p_{n,j} \leq 1$, $j = 0, 1, \dots, n$ and $\sum_{j=0}^{n} p_{n,j} = 1$. The j end points of the j new edges are chosen uniformly at random from the vertices of the generic n-clique.

Now, the j old connected vertices chosen, the new vertex and the j new edges form a $(j+1)$-clique. This new $(j+1)$-clique is a child of the generic n-clique and it is the only child at this step.

We see that any birth time the generic n-clique produces precisely one child. If $j = 0$, then this child is just one vertex. If $j = 1$, then this child is an edge. If $j = 2$, then this child is a triangle. If $j = n < N$, then this child is an $(n+1)$-clique. We underline, that we do not allow the birth of an $(N+1)$-clique.

The ancestor clique, the children cliques of the ancestor, the grandchildren cliques, etc. form an evolving network. When a clique dies, we do not delete it, but we consider it as a non-active individual not producing offspring.

An offspring j-clique is an individual of type j, so we shall denote it by subscript j. We shall denote by $\xi_{i,j}(t)$ the number of type j offspring of the type i generic object up to time t $(i, j = 1, 2, \dots, N)$. Then

$$\xi_i(t) = \sum_{j=1}^{i+1} \xi_{i,j}(t) \tag{1}$$

is the number of all offspring of the generic i-clique up to time t.

Let λ_i be the (random) life-length of the generic i-clique. We assume that the hazard rate of λ_i depends on the total number of offspring, so that

$$l_i(t) = b + c\xi_i(t)$$

with positive constants b and c. We can show that the survival function for an i-clique is

$$P(\lambda_i > t) = e^{-t(b+1)} e^{\frac{1-e^{-ct}}{c}}.$$

We can calculate the mean offspring number, that is $m_{i,j}(t) = E\xi_{i,j}(t)$ the expectation of the number of type j offspring of a type i team until time t.

To obtain asymptotic results, we suppose that our branching process is supercritical and satisfies some other reasonable conditions.

We obtain several limit theorems having the following shape.

Let n be fixed, $1 \leq n \leq N$. Let ${}_kT(t)$ denote the number of all n-cliques being born up to time t if the ancestor of the population was a k-clique, $k = 1, \dots, N$. Then

$$\lim_{t \to \infty} e^{-\alpha t} {}_kT(t) = {}_kW \frac{v_k u_n}{\alpha D(\alpha)}$$

almost surely for $k = 1, \dots, N$, where v_k, u_n, and $D(\alpha)$ are non-random and they are given by the parameters of the process, ${}_kW$ is an almost surely non-negative random variable, $E_k W = 1$, ${}_k W$ is a.s. positive on the event when the total number of offspring converges to infinity.

The proofs are based on known results of multi-type branching processes, see e.g. [6].

The mathematical theorems are supported by computer simulations, too.

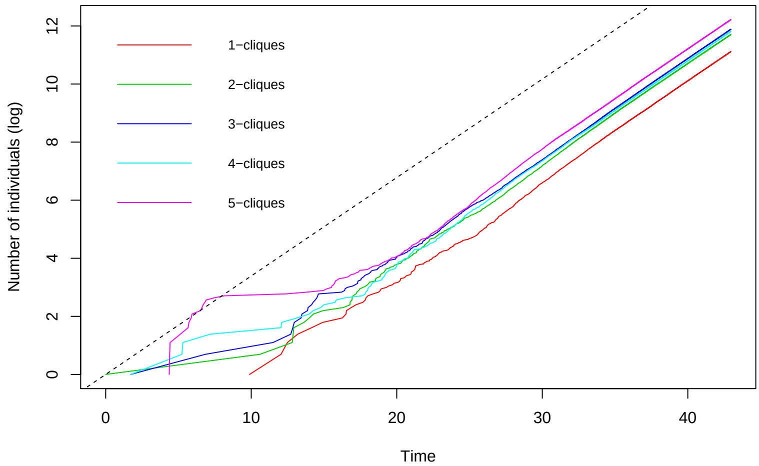

Figure 1. A simulation example of network growth with $N = 5$.

Figure 1 shows the number of n-cliques being born, $n = 1, 2, \dots, 5$. The dashed line shows the theoretical slope.

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