The Fritz-John condition system in Interval Branch and Bound method

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Abstract

The Interval Branch and Bound (IBB) method is a good choice when a rigorous solution is required. This method handles computational errors in calculations. Few IBB implementations use the Fritz-John (FJ) optimality conditions to eliminate non-optimal boxes in a constrained nonlinear programming problem. The FJ optimality condition means effectively a solution to an interval-valued system of equations. In the best case, the solution is an empty set if the interval box does not contain any optimum. In many cases, solving this system of equations is complex or fails. This problem can be caused by the fact that the interval box contains many solutions, or the defined system of equations contains unnecessary conditions, or the interval Gauss-Seidel method fails. These unsuccessful attempts have a negative outcome and only increase the computation time. In this study, we propose three modifications to reduce the runtime and the computational requirements demand of the Interval Branch and Bound method.

1. Introduction

This study focuses on solving the constrained nonlinear programming problems only with inequality and general bound constraints. We deal with the following nonlinear problem,

$$\begin{aligned} \underset{x \in \boldsymbol{x} \subseteq \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, \quad i = 1, \dots, m, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$ are continuously differentiable nonlinear functions and interval box $\boldsymbol{x} = [\underline{x}, \overline{x}]$ denotes a general bound constraint. For solving this problem, we use a guaranteed method, namely Interval Branch and Bound method (IBB).

2. Interval Branch and Bound

The Interval Arithmetic is a mathematical technique which helps to automatically account for computational errors (round-off and measurement errors). In this technique, we represent each value as an interval instead of a single number (integer, float) and we define all arithmetic operations and elementary functions on intervals. The interested readers can study the interval arithmetic definitions and basic notations in [3].

The IBB is based on the basic idea of the Branch and Bound method, where we split the search space into smaller regions. After, we replace the initial problem with smaller subproblems. Sometimes we can eliminate suboptimal or infeasible subproblems as we know bounds on the optimum or on the constraints. In the IBB method, we compute these bounds by Interval Arithmetic. In an IBB algorithm, there are five main steps: selection, bounding, discarding, division, and termination. These steps have to be specified for their implementation, and their choices can have a huge effect on the efficiency of the method. An introduction to the IBB method can be found in [4].

In IBB method, we can define many type of discarding tests: the midpoint, the cut-off tests, the monotonicity test in unconstrained case and the feasibility test, Karush-Kuhn-Tucker (KKT) [2] or Fritz-John [6] optimality tests in the constrained case. In this study, we focus only on the FJ optimality test, as it is more general than the KKT conditions.

3. Interval version of the Fritz-John condition

The Fritz John conditions are necessary conditions for a solution in nonlinear programming to be optimal. The Fritz John optimality conditions for a given point x are the $(2) - (3)$ system of equations, where $\mu_i \geq 0$ are the Lagrange multipliers. The straightforward extension of the Fritz John optimality conditions [1] for a given interval box $\boldsymbol{x}$ are the $(4) - (5)$ interval-valued system of equations, where $\boldsymbol{\mu_i}$ are the Lagrange multipliers from interval $[0, 1]$, $\boldsymbol{f}$, $\boldsymbol{g_i}$ are the inclusion functions for f, g_i and $\boldsymbol{\nabla f}, \boldsymbol{\nabla g_i}$ are the enclosures of the gradients for f, g_i. Solving an interval-valued system of equations, we can use the most common iterative method, namely the Interval Gauss-Seidel (IGS) method [5].

$$\mu_0 \nabla f(x) + \sum_{i=1}^{m} \mu_i \nabla g_i(x) = 0 \quad (2) \qquad \boldsymbol{\mu_0} \boldsymbol{\nabla f}(\boldsymbol{x}) + \sum_{i=1}^{m} \boldsymbol{\mu_i} \boldsymbol{\nabla g_i}(\boldsymbol{x}) = 0 \quad (4)$$

$$\mu_i g_i(x) = 0 \quad i = 1, \ldots, m \quad (3) \qquad \boldsymbol{\mu_i} \boldsymbol{g_i}(\boldsymbol{x}) = 0 \quad i = 1, \ldots, m, \quad (5)$$

4. Discussion

In this study, we focus on solving the Fritz-John optimality condition system. By solving the above presented interval-valued system of equations, we either discard the box, if the solution is an empty set. However, many times we either cannot remove any part of the box, or solving the Fritz-John system fails. This failure can be caused by the fact that the interval box is too large or the enclosures of the gradients contain zero. In the latter case we cannot divide by the interval which contains zero, thus the IGS method fails. Additionally, it also can happen that the defined system of equations contains unnecessary conditions, taking into account all the constraints. In general, trying to solve such an interval-valued system of equations has only negative outcome and only increases the computation time.
In this talk, we present three modifications for solving the Fritz-John condition system. First, we modify the IGS to handle the case where the interval $\boldsymbol{x}$ contains zero. In this case, we solve the system for the non-zero components, and split the other components using the extended division. In the second modification, we try to reduce the number of the iterative steps as much as possible. We use a method for estimating the initial bounds for the Lagrange multipliers $\boldsymbol{\mu_i}$. This is useful when we cannot reduce the Fritz-John condition system with the initial $\boldsymbol{\mu_i} = [0, 1]$ in one iteration. In the third modification, we introduce some preliminary criterion in the solution steps. We define some efficient criteria to either discard the box, or not to solve the interval-valued system of equations. With these modifications, we improve the runtime and computational requirements (number of function and gradient evaluation, the size of the working list) of the Interval Branch and Bound method.

References

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