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New Methods for Maximizing the Smallest Eigenvalue of Grounded Laplacian Matrix

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Abstract

The aim of this work is to maximize the smallest eigenvalue of the grounded Laplacian matrix which is the Laplacian matrix's $(n-k)\times(n-k)$ submatrix after k rows and associated columns have been removed. Thus, the problem is to choose optimally the rows and columns to be deleted. Motivated by the Gershgorin circle theorem, the degree centrality is used to select k nodes that would maximize the smallest eigenvalue. In addition, we have employed the vertex cover problem as an additional method of solving the problem. The efficiency of these approaches is demonstrated on real-world graphs.

1. Preliminaries

The Laplacian matrix (L) for a graph G with nodes set V and edges set E, $|V| = n$ and $|E| = m$, is defined as follows:

$$L_{ij} = \begin{cases} degree(i) & \text{if } i = j, \\ \text{-1} & \text{if } (i,j) \in E, \\ 0 & \text{otherwise} \end{cases}$$

Laplacian matrix is a symmetric positive semidefinite matrix, according to [1] its eigenvalues are non-negative, equal to the number of vertices, and less than or equal to twice the maximum vertex degree. As a result, $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. The grounded Laplacian matrix $L(S)$ is $(n-k)\times(n-k)$ submatrix obtained by deleting k rows and their corresponding columns from the Laplacian matrix L, where $S = \{V \setminus K\}$, $|K| = k$, $k > 0$, and $\lambda(S)$ is the smallest eigenvalue of $L(S)$. Finding $L(S)$ with the maximum possible $\lambda(S)$ is the problem to solve which has been shown to be an NP-hard [3].

2. Methodology

The first method that has been used is described in Algorithm 1. The idea is to rank the nodes according to their degree centrality, then remove the corresponding row and column from the Laplacian matrix. Note that this simple method is motivated by the famous Gerschgorin circle theorem.

Algorithm 1: Degree based Algorithm

1. $deg\_cen = sort(degree(V))$
2. **for** $i \in 1 \to k$ **do**
3. $\quad remove(L(deg\_cen[i]))$
4. $compute(min\_eigen(L))$

The second method is maximum k vertex cover, which is based on the vertex cover problem [2], with the difference, that in the k vertex cover search for k nodes that incident to the maximum number of edges of the graph rather than the minimum number of nodes that each edge in the graph is incident to. The linear program of the problem is defined as follows:

$$\max \sum_{j \in V} y_j$$

$$\sum_{i \in V} x_i = k$$

$$y_j \leq \sum_{(j,i) \in E} x_i$$

$$k \in \mathbb{N}, \quad x_i, y_i \in \{0,1\}, \quad i = 1, \ldots, n.$$

Note that the variables x_i stand in for the k nodes that incident to the maximum number of edges in the graph, where y_i represents the objective to maximize, y_i is 1 if at least one of its adjacent nodes, x_i, is 1. Once we have the nodes, we delete the rows and columns that correspond to them from the Laplacian matrix and then determine its smallest eigenvalue.

3. Preliminary results

To demonstrate and compare the efficiency of the proposed methods, we present some numerical results on real-world graphs. Results obtained by utilizing the degree centrality method, where nodes are sorted ascending (blue) and descending (orange), are shown in Figure 1. Note that we fix k to be 30. The results of the vertex cover problem on several real-world graphs with various values of k are shown in Figure 2.

Further results along with more elaborated methods will be presented at the conference. The aim is to develop additional approaches to solve the problem and compare the results with those of Wang et al.[3].

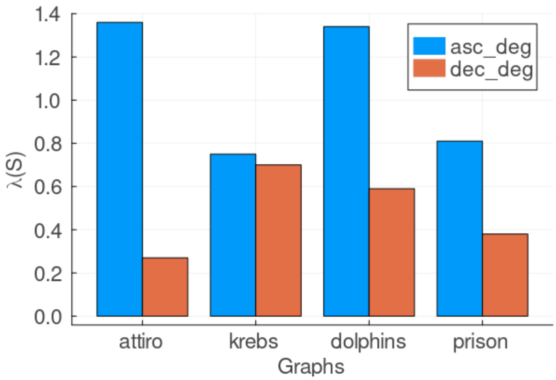

Figure 1. $\lambda(S)$ with $k = 30$ by using degree centrality method

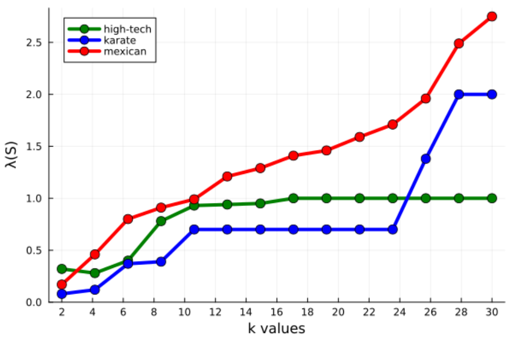

Figure 2. $\lambda(S)$ with different k by using vertex cover

References

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