Using Extended Resolution to Represent Strongly Connected Components of Directed Graphs

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Abstract

There could be some interesting links between directed graphs and SAT problems [2]. On the one hand, directed graphs could represent many types of objects, but in many cases, it is not so straightforward how and what type of representation leads to some advantages. Since, this problem generally seems to be very difficult, we work on a related one: represent a directed graph as a SAT problem. Here we have several models [4]. Each of them has the following property: If the represented directed graph is strongly connected then its SAT representation has only two solutions, the one where all variables are true, and the one where all variables are false. SAT problems of this type are called Black-and-White SAT [1]. In this paper we study those directed graphs which consist of not only one strongly connected component (SCC), but more. In this work we show that if a directed graph consists of two components, A and B, and there is an edge from A to B, then the corresponding SAT representation has a third solution which is $\neg A$ union B. We generalize this lemma for more complex graphs. Furthermore, we study the question how to represent an SCC by one Boolean variable to keep the previous properties. We found out that extended resolution [3, 6] is a suitable tool for that. To represent the SCC which consists of only two vertices a and b, we have to add to its model the following formula: $a \wedge b$ equals x, i.e., the following clauses: $\neg a \vee \neg b \vee x$, $\neg x \vee a$, and $\neg x \vee b$. This is the classical example of extended resolution, where x is a new variable. Although, the original problem is

still very difficult, this work helps us to understand better what extended resolution means, and how to represent extended resolution graphically. Most specially, how to represent extended resolution in a resolvable network [5].

The strong model, our first model, was defined formally in our first paper [1]. We recall its definition.

Let $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ be a communication graph, then the strong model of $\mathcal{D}$ is denoted by $\mathcal{SM}$, and defined as follows:

$$\mathcal{SM} := \{\{\neg a,\ b\} \mid (a, b) \in \mathcal{E}\}.$$

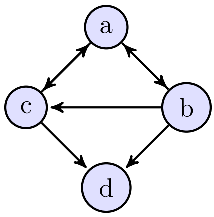

Figure 1. A communication graph with 4 vertices, 3 cycles.

As an example we show the strong model of the communication graph of Figure 1:

$$\begin{aligned} \mathcal{SM} = \{&\{\neg a, b\}, \{\neg a, c\}, \{\neg b, a\}, \{\neg b, c\}, \\ &\{\neg b, d\}, \{\neg c, a\}, \{\neg c, d\}\}. \end{aligned} \tag{1}$$

Note that since the communication graph in Figure 1 is not strongly connected, its strong model (1) is not a Black-and-White SAT problem. For example $\{\neg a, \neg b, \neg c, d\}$ is a solution of it.

Note, that this communication graph consists of two Strongly Connected Components (SCCs), which are: $\{a, b, c\}$ and $\{d\}$. Furthermore, there are edges from the first SCC to the second one. Hence, its model has one more solution: $\{\neg a, \neg b, \neg c, d\}$.

Now we can use extended resolution to introduce x instead of the first SCC. Thus, let x denote $a \wedge b \wedge c$. So, let x be the formula $a \wedge b \wedge c$. In this case we have to add following clauses to the strong model of the graph: $\neg a \vee \neg b \vee \neg c \vee x$, $\neg x \vee a$, $\neg x \vee b$, and $\neg x \vee c$. It is easy to check that the new model still has only 3 solutions: $\{\neg a, \neg b, \neg c, \neg x, d\}$, $\{a, b, c, x, d\}$, and $\{\neg a, \neg b, \neg c, \neg x, \neg d\}$. This means that the first SCC can be substituted by the single variable x which greatly simplifies the graph and also its model.

We know that resolvable networks can represent any SAT problem [5]. Now, some very interesting questions are arising. How to generalize the notion of SCC for resolvable networks? How can we recognize extended resolution? Especially, how can we recognize the new variable of extended resolution in a resolvable network?

References

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