13th International Conference on Applied Informatics
Eszterházy Károly Catholic University
Eger, Hungary, February 19–21, 2026

Solving differential-functional equations with computer

Attila Házy[a]

[a]Eszterházy Károly Catholic University
hazy.attila@uni-eszterhazy.hu

Abstract

We present an algorithm and a program (developed in the computer algebra system Maple), which determines the solution of linear functional equations belonging to the class

$$h_0(x,y)f_0(g_0(x,y)) + \cdots + h_n(x,y)f_n(g_n(x,y)) = F(x,y)$$

where $n \in \mathbb{N}$, $g_0, g_1, \ldots, g_n, h_0, h_1, \ldots, h_n$, and F are given real valued functions on an open set $\Omega \subset \mathbb{R}^2$, furthermore $f_0, f_1 \ldots, f_n$ are unknown functions.

Applying the results of Páles [10] and Házy [8] we get a differential-functional equation in one of unknown function for $f_1, f_2, \ldots, f_n$. The solutions of this equation are the same as that of an ordinary differential equation (under some assumptions).

In the theory of functional equations there exist few methods is solving a wider class of functional equations. In several cases, with the help of the computer algebra system Maple V, which enables us to perform the tedious computations, we completely describe the general solutions of these equations. See for example L. Székelyhidi's results ([11]), the papers of Sz. Baják and Zs. Páles ([1–3]), S. Czirbusz ([5]), A. Házy ([8]), A. Gilányi [4, 6, 7] and C. P. Okeke, T. Szostok ([9]).

An algorithm of the reduction of linear two variable functional equations

$$h_0(x,y)f_0(g_0(x,y)) + \cdots + h_n(x,y)f_n(g_n(x,y)) = F(x,y), \tag{1}$$

to differential-functional equations has been found by Zs. Páles [10]. Here $g_0, g_1, \ldots, g_n, h_0, h_1, \ldots, h_n$ and F are given real valued functions on an open set $\Omega \subset \mathbb{R}^2$ and $f_0, f_1, \ldots, f_n$ are unknown functions.

Using the result of Páles [10], we get a partial differential operator which eliminate all unknown function from the equation, except the first. That is, applying a linear differential operator $\mathbf{D}$ to the equation (1), we get the following linear differential-functional equation:

$$\mathbf{D}[h_0(x,y)f_0(g_0(x,y))] = \mathbf{D}F(x,y), \qquad (x,y) \in \Omega.$$

Based on this result, we get an algorithm to transform the linear two-variable functional equation to differential-functional equation for f_0. The detailed description can be found in the paper of Páles [10] and Házy [8].
At the end of this algorithm we get an equation for f_0. We showed in [8] that there exists an ordinary differential equation (under some assumptions) whose solutions are the same as the that of obtained differential-functional equation. It is shown, that if we consider the equation

$$l_k(x,y)f^{(k)}(g(x,y)) + \ldots + l_0(x,y)f(g(x,y)) = L(x,y), \tag{2}$$

for all $(x,y) \in \Omega$, where $l_0, l_1, \ldots, l_k, g$ and L are given real valued analytic functions on Ω, furthermore f is an unknown real function on $g(\Omega)$, then there exists a differential equation

$$f^{(m)}(t) + \sum_{i=0}^{m-1} K_i(t)f^{(i)}(t) = K(t), \tag{3}$$

whose $(k+1)$-times differentiable solutions coincide with that of (2). This equation (3) reduces to an ordinary differential equation with respect to f.
To illustrate this method, we present an examples.

Example 1. Consider the functional equation

$$(x+y)f(x+y) - yg(x) - h(y) = x+y. \tag{4}$$

Applying the differential operator $\mathbf{D} = \partial_x - y\partial_x\partial_y$ to this equation we get the following second order differential-functional equation for f:

$$f(x+y) + (x-y)f'(x+y) - (xy+y^2)f''(x+y) = 1. \tag{5}$$

Using the differential operator $\mathbf{D_0} = \partial_x - \partial_y$ to (5) then we get the equation

$$(x+y)f''(x+y) + 2f'(x+y) = 0. \tag{6}$$

The order of this equation is also 2, that is the order is not increase. If multiply the equation (6) by $(xy+y^2)$, and the equation (5) by $(x+y)$ and adding up these equations, we obtain the following first order equation:

$$(x+y)^2 f'(x+y) + (x+y)f(x+y) = x+y. \tag{7}$$

The solutions of this equation (7) are the same as that of (5). It is easy to see (with substitution $t = x + y$) we get

$$t^2 f'(t) + tf(t) = t,$$

which implies that

$$f(t) = 1 + \frac{c}{t}.$$

Substituting this function f back to the equation (4), we get

$$(x+y)\left(1 + \frac{c}{x+y}\right) - yg(x) - h(y) = x + y,$$

that is

$$yg(x) + h(y) = c,$$

which implies that there exists constant d such that

$$g(x) = d, \qquad h(y) = c - dy.$$

References

[1] S. Baják, Z. Páles: *Computer aided solution of the invariance equation for two-variable Gini means*, English, Comput. Math. Appl. 58.2 (2009), pp. 334–340, issn: 0898-1221, doi: 10.1016/j.camwa.2009.03.107.

[2] S. Baják, Z. Páles: *Computer aided solution of the invariance equation for two-variable Stolarsky means*, English, Appl. Math. Comput. 216.11 (2010), pp. 3219–3227, issn: 0096-3003, doi: 10.1016/j.amc.2010.04.046.

[3] S. Baják, Z. Páles: *Solving invariance equations involving homogeneous means with the help of computer*, English, Appl. Math. Comput. 219.11 (2013), pp. 6297–6315, issn: 0096-3003, doi: 10.1016/j.amc.2012.11.032, url: hdl.handle.net/2437/171459.

[4] G. G. Borus, A. Gilányi: *Computer assisted solution of systems of two variable linear functional equations*, English, Aequationes Math. 94.4 (2020), pp. 723–736, issn: 0001-9054, doi: 10.1007/s00010-020-00736-z.

[5] S. Czirbusz: *Testing regularity of functional equations with computer*, English, Aequationes Math. 84.3 (2012), pp. 271–283, issn: 0001-9054, doi: 10.1007/s00010-012-0157-7.

[6] A. Gilányi: *Solving linear functional equations with computer*, English, Math. Pannonica 9.1 (1998), pp. 57–70, issn: 0865-2090, url: https://eudml.org/doc/231137.

[7] A. Gilányi, L. N. To: *On a computer program determining types of functional equations*, English, Aequationes Math. 99.6 (2025), pp. 2565–2579, issn: 0001-9054, doi: 10.1007/s00010-025-01184-3.

[8] A. Házy: *Solving linear two variable functional equations with computer*, English, Aequationes Math. 67.1-2 (2004), pp. 47–62, issn: 0001-9054, doi: 10.1007/s00010-003-2703-9.

[9] C. P. Okeke, T. Szostok: *Solving linear functional inequalities with a computer*, English, Aequationes Math. 99.6 (2025), pp. 2871–2891, issn: 0001-9054, doi: 10.1007/s00010-025-01218-w.

[10] Z. Páles: *On reduction of linear two variable functional equations to differential equations without substitutions*, English, Aequationes Math. 43.2-3 (1992), pp. 236–247, issn: 0001-9054, doi: 10.1007/BF01835706, url: https://eudml.org/doc/137467.

[11] L. Székelyhidi: *On a class of a linear functional equations*, English, Publ. Math. Debr. 29 (1982), pp. 19–28, issn: 0033-3883, doi: 10.5486/PMD.1982.29.1-2.02.