

Convergence to generalized permutons

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Abstract

The asymptotic behaviour of sequences of permutations was studied in [5]. It was proved, that any sequence of permutations $\sigma_k = (\sigma_k(1), \sigma_k(2), \dots, \sigma_k(k))$ which is convergent in the sense of permutation densities as $k \rightarrow \infty$, has a limit object. The limit object is a probability measure on the unit square such that both marginals are uniform distributions on the unit interval. In the original paper [5], the limit object is called limit permutation, but its usual name is copula (see [4]) and nowadays it is often called permuton (see [3]). In [5] it was also shown, that any permuton is the limit of some convergent permutation sequence. The study in [5] was inspired by well-known results on limits of graph sequences, see e.g. [6]. After the publication of the paper [5], several studies were devoted to the asymptotic behaviour of permutation sequences, see e.g. [1].

In [2], we extended the results of [5] to ordered selections in the following way. We select m elements out of the set $\{1, 2, \dots, n\}$ without replacements and assume that the order of the selection matters. Denote the selected numbers by $\nu = \nu_{n,m} = (\nu(1), \nu(2), \dots, \nu(m))$. We call $\nu_{n,m}$ an (n, m) -permutation. We study the limiting behaviour of $\nu_{n,m}$ as $n, m \rightarrow \infty$ so that $\frac{m}{n} \rightarrow \lambda \in (0, 1]$. In [5], it was shown that the permutation densities determine the convergence of a sequence of usual permutations. However, we can see that the permutation densities alone do not describe the asymptotic stability of a sequence of (n, m) -permutations. To this end, we need also the distribution of the selected ‘places’ in the sequence $\{1, 2, \dots, n\}$. Using together the permutation densities and the distribution of the selected ‘places’, we can introduce a convergence notion which we denote by $\nu_{n,m} \xrightarrow{t}$. Our goal is to describe the limit object. Because $n \rightarrow \infty$, the limit object is outside of the set of permutations. Following the ideas of [5], we can identify the

limit object μ as a probability measure which we call λ -permuton.

The probability measure μ is called a λ -permuton, if for a certain λ , $0 < \lambda \leq 1$, μ is concentrated on $[0, \lambda] \times [0, 1]$, its first marginal distribution function is uniform on $[0, \lambda]$, and the second marginal distribution function is continuous and its slope is at most $1/\lambda$. For a fixed λ , let \mathcal{W}_λ denote the set of all λ -permutons. Let $\mathcal{W} = \cup_{\lambda \in (0, 1]} \mathcal{W}_\lambda$. The elements of \mathcal{W} are called generalized permutons.

In the set \mathcal{W} of generalized permutons, we introduce the notion of convergence in the sense of permutation densities which we denote by $\mu_n \xrightarrow{t} \mu$. We show that for generalized permutons, this convergence is equivalent to the usual convergence of probability measures and as well as to the convergence in the sense of rectangular distance and also in the sense of L_∞ distance of the distribution functions.

The (n, m) -permutations can be embedded into the space of generalized permutons. In this way, we can prove that any convergent sequence of (n, m) -permutations has a limit, which is a generalized permuton. Without going into the details of the precise definitions, we can present our main result:

Let $\nu_n = \nu_{n,m}$ be a convergent sequence of (n, m) -permutations: $\nu_n \xrightarrow{t}$. Let μ_n be the generalized permuton corresponding to ν_n , $n = 1, 2, \dots$. Then the sequence μ_n is convergent, i.e. there exists a generalized permuton μ such that $\mu_n \xrightarrow{t} \mu$.

So we can consider the generalized permuton μ as the limit of the sequence of the (n, m) -permutations, so we can write $\nu_n \xrightarrow{t} \mu$.

For the proof of our theorem, we use appropriate modifications of the methods of [5].

Our next theorem shows that any generalized permuton is the limit of a sequence of (n, m) -permutations:

Let μ be a generalized permuton. Then there exists a sequence $\nu_n = \nu_{n,m_n}$ of (n, m_n) -permutations, such that $\nu_n \xrightarrow{t} \mu$ as $n \rightarrow \infty$.

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