

A Family of Discrete-time Network Evolution Models Based on k -interactions

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Abstract

We study a parametrized family of discrete time network evolution models. The evolution of the graph is based on constructions and deletions of k -cliques. Using martingale theory, we prove an almost sure limit theorem for the number of vertices, then show its asymptotic normality. We obtain an almost sure limit theorem for the degree of a fixed vertex. We also present an asymptotic normality result for the degree of a fixed vertex. A functional limit theorem is obtained for the number of vertices, and a multidimensional functional limit theorem is proved for the joint behaviour of the degrees of several fixed vertices. For the proofs, we apply martingale functional limit theorems. We also offer simulation results supporting our theorems.

1. Introduction

Network theory is important both for real-life applications and theoretical research. It studies general properties of networks and offers models and methods to understand their evolution. Well-known large networks are e.g. the World Wide Web, the Internet, metabolic networks, and social networks. It has applications in logistics, electrical engineering, biology, economics, ecology, public health, sociology and many other fields. One can find several general facts on network theory in the book [2]. A random graph can describe an evolving network. The vertices of the graph are the nodes of the network and the edges of the graph are the connections among the nodes. Our aim is to introduce a new network evolution procedure and find its basic properties.

[1] introduced a random graph evolution model with moderate edge density. Their model is the following. The starting graph is an empty one of size 2. At each step, two vertices are chosen uniformly at random. If the two vertices chosen are not connected, we connect them with 1 edge. If the two vertices are connected, we delete the connecting edge, add a new vertex to the graph, and connect the new vertex to both of the selected vertices. The result of this procedure is an evolving graph containing n edges after the n^{th} step. In [1], several asymptotic theorems are proved for the number of vertices and the degree of a fixed vertex. Also in [1], a short overview is presented about the edge densities of some well-known random graph models.

We study the following extension of the model of [1]. Instead of connections of two vertices, we consider connections of k vertices, where $k \geq 2$ is a fixed integer. So the main ingredients of our model are the k -cliques. The evolution of our graph is based on constructions and deletions of k -cliques. A k -clique is a sub-graph containing k vertices and any two different vertices are connected by 1 edge. When we form a k -clique, then we draw $\binom{k}{2}$ new edges among k vertices, and we add this new clique to the list of k -cliques.

The initial graph at time $n = 0$ contains k vertices and no one edge. In the first step i.e. when the time is $n = 1$, we connect the k vertices to obtain a single k -clique. Then, in each step, we choose k vertices uniformly at random from the existing vertices. If they do not form a k -clique, then we construct a new k -clique on these vertices. In the other case, when the sub-graph consisting of the k vertices chosen is a k -clique, then that k -clique is deleted. Then a new vertex is added to the graph and two new k -cliques are created.

2. Some Results

Let V_n denote the number of vertices in the model after n steps. In the first theorem we prove that the magnitude of V_n is $n^{\frac{2}{k+1}}$ and V_n is asymptotically normal. Let us denote the normal distribution with mean m and variance σ^2 by $\mathcal{N}(m, \sigma^2)$.

Theorem 1. *As $n \rightarrow \infty$, the following almost sure convergence holds for the number of vertices in the graph after n steps:*

$$\frac{V_n}{\left[\frac{(k+1)!}{2}\right]^{\frac{1}{k+1}} n^{\frac{2}{k+1}}} \rightarrow 1. \tag{1}$$

Furthermore, we have

$$\frac{1}{n^{\frac{1}{k+1}}} \left(V_n - \left[\frac{(k+1)!}{2}\right]^{\frac{1}{k+1}} n^{\frac{2}{k+1}} \right) \Rightarrow \mathcal{N}\left(0, \frac{1}{2k+1} \left[\frac{(k+1)!}{2}\right]^{\frac{1}{k+1}}\right) \tag{2}$$

as $n \rightarrow \infty$, where \Rightarrow denotes convergence in distribution.

Let us use the notation $d_n(v)$ for the degree of the vertex v at time n . In the following theorem, we calculate the asymptotic degree of the vertices.

Theorem 2. *For the degree of vertices with labels $v = 1, 2, \dots$ we have*

$$d_n(v) \sim k(k+1) \left[\frac{2}{(k+1)!} \right]^{\frac{1}{k+1}} n^{\frac{k-1}{k+1}} \quad (3)$$

almost surely as $n \rightarrow \infty$.

We have also carried out simulation studies to support our theorems. We also have a functional limit theorem for the number of vertices and multidimensional functional limit results for the joint distribution of the degrees of several fixed vertices. Those theorems are more general, and they also imply that the joint distribution of the degrees of vertices is asymptotically normal and the degrees of different vertices are asymptotically independent.

The detailed description of our results can be found in [3].

References

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